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Coherent Transport and Dynamical Entropy for Fermionic Systems

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This paper consists of two parts. First we set up a general scheme of local traps in a homogeneous deterministic quantum system. The current of particles caught by the trap is linked to the dynamical behaviour of the trap states. In this way, transport properties in a homogeneous system are related to spectral properties of a coherent dynamics. Next we apply the scheme to a system of Fermions in the one-particle approximation. We obtain in particular lower bounds for the dynamical entropy in terms of the current induced by the trap.

KEY WORDS: Coherent transport; scaling exponents; Fermion systems in one-particle approximation; dynamical entropy.

1. INTRODUCTION

In this paper, we are interested in time scaling properties of propagation by coherent quantum dynamics. Non-trivial behaviour of a single-particle Hamiltonian can lead to a description of anomalous diffusion of electrons in solids.[(1)] This behaviour becomes apparent through the scaling of the spreading of one-particle wave functions with respect to time. We shall here adopt another approach: we introduce a localised trap in an infinite system and study the time behaviour of the current of particles falling in the trap. Applied to systems of Fermions in the non-interacting approximation, we obtain a lower bound on the dynamical entropy in terms of this current.

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The trap states will be described by a collection of wave functions and we relate the current to the dynamics of these states. In particular, we show that an absolutely continuous spectrum produces a non-zero asymptotic current. A singular spectrum will lead to asymptotically vanishing currents possibly characterised by a dynamical exponent.

A number of related topics and models have been considered in the literature, mostly for the case of a continuous time evolution. The occurrence of singular continuous spectra as a source of anomalous diffusion is caused either by randomness in the Hamiltonian or by aperiodicity. We shall however not be concerned by producing such models but rather link scaling properties of dynamical entropy to assumed spectral properties of a discrete dynamics.

Coherent transport in quantum systems is being studied by using reservoirs as drivers. A number of delicate questions arise in this context with respect to the thermodynamic limit. Anomalous transport due to spatial randomness in the dynamics seems to occur.[2]

Strongly chaotic classical or quantum dynamical systems generate entropy at a non-zero asymptotic rate: the dynamical entropy. In the classical case, the sum of the positive Lyapunov exponents is a bound for the entropy (Ruelle's inequality) and equality is reached for sufficiently smooth systems (Pesin's theorem). For quantum dynamical systems several entropies have been introduced such as the CNT construction based on a coupling with a classical system and the ALF construction that relies on POVM's (operational partitions of unity). In order to obtain a non-zero entropy an absolutely continuous dynamical spectrum is needed, at least for Fermion systems in the one-particle approximation.[3] In open classical systems, the escape rate formalism links the escape exponent from an unstable repeller to diffusive transport. The escape rate is given by the missing exponents in the entropy for motion on the repeller.[4]

There are however many mixing dynamical systems with less pronounced randomising properties which are not given in terms of exponents or rates. Such dynamics may lead to a sublinear scaling for the total dynamical entropy.[5]

The aim of this work is to establish a lower bound for the entropy in terms of dynamical exponents of a localised trap in an infinite system both in the regular and the anomalous case.

As a motivation we provide in Section 2 a few simple examples of the use of a trap in classical dynamics. Obviously there is a different physical mechanism at work with possibly similar macroscopic effects but our aim is to show that the time behaviour of the current at the edge of a localised trap encodes relevant information about the transport properties of the dynamics. Because of the locality of the trap, it is possible to deal immediately with an infinite system and to avoid using a delicate large volume limit of boundary conditions. Section 3 deals with localised traps in unitary Hilbert space dynamics and we study in Section 4 the dynamical entropy for non-interacting Fermions. We obtain in particular a lower bound for the entropy in terms of the current of particles falling in a trap.

2. TRAPS IN A CLASSICAL CONTEXT

Consider first the simplest classical counterpart of a trap absorbing free electrons moving with velocities below that corresponding to a given Fermi level. This is a system of classical particles homogeneously distributed in space and with uniform velocity distribution below a maximal one. The state of such a system is a uniform mixture of spatially homogeneous states with fixed velocity. The number of particles per time unit caught in a localised trap is constant in time and essentially determined by the average cross-section of the trap.

Next we consider the model of the diffusion equation in one dimension with a trap at the origin. We have to find the solution of the equation

$$\frac{\partial n}{\partial t} = D\,\frac{\partial^2 n}{\partial x^2}$$

for $x > 0$ and $t > 0$ with boundary condition $n(0, t) = 0$ and initial condition $n(x, 0) = 1$. In this equation D is the diffusion constant and $n(x, t)$ represents the particle density at time t and place x and the particle current is given by Fick's law

$$j(x, t) = -D\,\frac{\partial}{\partial x}\,n(x, t).$$

The solution of the diffusion equation reads

$$n(x, t) = \frac{2}{\pi}\int_0^\infty \mathrm{d}k\,\frac{\sin(kx)}{k}\,\mathrm{e}^{-Dk^2 t}.$$

Therefore, the current at $x = 0$ is

$$j(t) = -\frac{2D}{\pi}\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-Dk^2 t} = -\frac{\sqrt{D}}{\sqrt{\pi t}}.$$

We recover hereby the usual exponent for diffusion.

The third example is a simple random walk in one dimension with the site zero absorbing the walker. Let $p(x, t)$ denote the probability that the walker reaches the origin for the first time at time t starting out at site x.

We may assume that $x \in \mathbb{N}$. The probabilities p are determined by the recursion relation

- $p(0, 0) = 1$ and $p(0, t) = 0$ for $t > 0$
- $p(x, t) = 0$ whenever $x > t$
- $p(x, t) = \frac{1}{2}(p(x-1, t-1) + p(x+1, t-1))$.

It can be checked that the solution is given by

$$p(x, t) = \begin{cases} \dfrac{1}{2^t}\left\{\begin{pmatrix} t-1 \\ (t-x)/2 \end{pmatrix} - \begin{pmatrix} t-1 \\ (t-x-2)/2 \end{pmatrix}\right\} & t-x \text{ even integer} \\ 0 & \text{else.} \end{cases}$$

The current at time t is then given by

$$J(t) = \sum_{x=1}^{t} p(x, t) = \frac{1}{2^t}\begin{pmatrix} t-1 \\ [(t-1)/2] \end{pmatrix} \sim \frac{1}{\sqrt{2\pi t}}.$$

In this formula, $[x]$ denotes the largest integer less or equal to x.

Finally, a random walk on $\mathbb{Z}^3$ with a trapping set A leads to a total current $J_A(t) = \mathrm{d}N_A(t)/\mathrm{d}t$ where $N_A(t)$ is the total number of particles trapped by the set A up to time t. It turns out that

$$J_A(t) \sim C(A) + 2(2\pi)^{-3/2}\, C(A)^2\, t^{-1/2},$$

where $C(A)$ is the capacity of the set A.[6] Again, the behaviour of the current returns the relevant information on the transport properties of the system.

The remainder of this paper will deal with reversible quantum dynamics. We shall first study in Section 3 the connection between the current at a local trap and the spectral properties of the evolution. In Section 4 we shall concentrate on a simplified model of an infinitely extended Fermion system and relate the current at a trap with the dynamical entropy of the system. It is necessary to deal with infinite quantum systems as the quantizations of conservative classical dynamical systems with compact phase spaces have an almost periodical time evolution. This not only prohibits true ergodic behaviour but even puts a finite upper bound on the total entropy.

Still, in the classical context, there is a vast literature on diffusive phenomena for classical deterministic models such as the Lorentz gas or the multibaker maps. One connects in this context diffusive characteristics, such as the escape rate, with chaotic characteristics of the system, such as Kolmogorov–Sinai entropy (rate) and Lyapunov exponents. Rather than looking at localised traps, as we propose, one considers particles escaping or entering the system at infinity. Recent results are reviewed in refs. 7 and 8.

The lower bound for the entropy in terms of the current that we shall obtain remains valid for systems with an abnormal behaviour, i.e., systems for which the entropy scales sublinearly in time. This has also, at least numerically, a classical counterpart, e.g., a Lorentz gas with diamond-shaped scatterers, see ref. 9.

3. TRAPS IN HILBERT SPACE

We first consider an abstract model of a trap that absorbs particles. An explicit connection with a model of Fermions will be presented in Section 4. The basic ingredients of our description are an infinite dimensional Hilbert space $\mathfrak{H}$, a unitary operator U on $\mathfrak{H}$ which specifies the evolution during a single time step, and a non-negative operator A less or equal than $\mathbb{1}$ which describes the effect of the trap. We shall assume that the trap is local in the sense that it is of finite rank d. Writing its eigenvalue decomposition

$$A = \sum_{j=1}^{d} p_j \, |\psi_j\rangle\langle\psi_j|, \qquad 0 \leqslant p_j \leqslant 1$$

we can think of p_j as the probability that a particle in the state $|\psi_j\rangle$ is absorbed by the trap when hitting the trap once. A value of p_j to 1 means that the trap captures particles very efficiently in the state $|\psi_j\rangle$.

Starting out with a density $0 \leqslant \rho \leqslant \mathbb{1}$, the density after one time step and hitting once the trap becomes

$$(\mathbb{1} - A)\, U\rho U^*(\mathbb{1} - A) = (TU)\, \rho(TU)^*$$

with $T := \mathbb{1} - A$. Assuming a uniform initial distribution, i.e., ρ a scalar multiple of $\mathbb{1}$, the number of particles absorbed up to time t by the trap is proportional to

$$N_A(t) := \mathrm{Tr}(\mathbb{1} - (TU)^t\, (U^*T)^t). \tag{1}$$

The operator $(TU)^t\,(U^*T)^t$ is a finite rank perturbation of $\mathbb{1}$, therefore formula (1) makes sense.

We shall always assume that eventually infinitely many particles are absorbed by the trap. This will certainly be the case if the dynamics has reasonable randomising properties. We are in particular interested in the scaling behaviour of $N_A(t)$ with time, i.e., in the exponent γ governing the asymptotics of N_A

$$N_A(t) \sim t^{1-\gamma}, \qquad t \text{ large.}$$

The exponent γ is non-negative as the growth of N_A is at most linear in time and cannot exceed the value 1 by our assumption $\lim_{t \to \infty} N_A(t) = \infty$. We expect this scaling to be related to the spectral properties of U and to be independent of the size of the trap. Using the telescopic formula

$$\mathbb{1} - (TU)^t (U^*T)^t = \sum_{s=0}^{t-1} (TU)^s (\mathbb{1} - T^2)(U^*T)^s,$$

we rewrite for $t \geqslant 1$ $N_A(t)$ in terms of a current J_A

$$N_A(t) = \sum_{s=1}^{t} J_A(s)$$

with

$$\begin{aligned} J_A(t) &:= N_A(t) - N_A(t-1) \\ &= \operatorname{Tr}\{(\mathbb{1} - (TU)^t (U^*T)^t) - (\mathbb{1} - (TU)^{t-1} (U^*T)^{t-1})\} \\ &= \operatorname{Tr}(TU)^{t-1} (\mathbb{1} - T^2)(U^*T)^{t-1}. \end{aligned} \tag{2}$$

The expected scaling behaviour of J_A is then

$$J_A(t) \sim t^{-\gamma}, \qquad t \text{ large.}$$

The actual analysis will be performed for the simple case where A is a one-dimensional projector $|\varphi\rangle\langle\varphi|$ with φ a normalised vector in $\mathfrak{H}$ and we shall henceforth drop the subscripts of N and J. In this case, N is fully determined by the probability measure

$$\mu(\mathrm{d}\theta) := \|E(\mathrm{d}\theta)\, \varphi\|^2 \tag{3}$$

on the unit circle S^1 where E is the spectral measure of U

$$U = \int_{S^1} \mathrm{e}^{-i\theta}\, E(\mathrm{d}\theta).$$

The expression (2) for the current now becomes

$$J(t) = \|(\mathbb{1} - P_{t-1}) \cdots (\mathbb{1} - P_1)\, \varphi\|^2, \qquad t > 1 \tag{4}$$

with P_t the orthogonal projector on $U^{-t}\varphi$. For consistency we must put $J(1) = 1$. As $(\mathbb{1} - P_t)$ is a contraction, J is a monotonically decreasing function and therefore

$$J_\infty := \lim_{s \to \infty} J(s)$$

exists. The behaviour of the current provides information on the transport properties of the system. As we don't have in our general description a notion of position operator, allowing a definition of ballistic or diffusive motion in terms of spatial spreadings of wave functions, we shall rather concentrate on the relation between the current and the randomising properties of the dynamics which are quantified by the dynamical entropy. This is, within the context of non-interacting Fermion systems, the subject of Section 4. In this section we shall study the asymptotic current in terms of the dynamical properties of the trap states for the simple one-state trap. The analysis could be extended to more involved traps that have a spatial structure. The more complicated behaviour of the subsequent current could then be studied as in the case of the 3D classical random walk, leading possibly to quantum capacities.

In order to relate the measure μ with J_∞, we introduce for $|z| < 1$ the function

$$G(z) := \sum_{s=1}^{\infty} z^s \, \mu\hat{}(s), \tag{5}$$

where $\mu\hat{}$ is the Fourier transform of μ

$$\mu\hat{}(t) := \int_{S^1} \mu(\mathrm{d}\theta)\, \mathrm{e}^{-it\theta}, \qquad t \in \mathbb{Z}. \tag{6}$$

Expressing G in terms of μ, we find

$$G(z) = \int_{S^1} \mu(\mathrm{d}\theta) \frac{z}{\mathrm{e}^{i\theta} - z}.$$

Obviously, G is analytic in the open unit disc and we shall be concerned with its value on the boundary of the disc. Writing $z = r\mathrm{e}^{i\eta}$ with $0 \leqslant r < 1$, a direct computation shows that

$$1 + 2\, \Re\mathrm{e}\, G(r\mathrm{e}^{i\eta}) = \int_{S^1} \mu(\mathrm{d}\theta) \frac{1 - r^2}{1 + r^2 - 2r\cos(\eta - \theta)}. \tag{7}$$

The function

$$\delta_r(\theta) := \frac{1-r^2}{1+r^2-2r\cos\theta},$$

is the Poisson kernel and the δ_r are a δ-convergent sequence of smooth, positive, normalised functions

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \delta_r(\theta) = 1$$

and

$$f(\eta) = \lim_{r\uparrow 1} \frac{1}{2\pi}\int_{S^1} \mathrm{d}\theta\, f(\theta)\, \delta_r(\theta-\eta) \quad \text{a.e.} \tag{8}$$

for any integrable function f on the unit circle. By a.e. we shall always mean almost everywhere with respect to the Lebesgue measure.

The imaginary part of G is given by

$$\mathfrak{Im}\, G(r\mathrm{e}^{i\eta}) = \int_{S^1} \mu(\mathrm{d}\theta)\, \frac{r\sin(\eta-\theta)}{1+r^2-2r\cos(\eta-\theta)}.$$

When r tends to 1, we obtain the Hilbert transform of μ[10, 11]

$$\begin{aligned}(\mathscr{H}\mu)(\eta) &= \lim_{r\uparrow 1}\int_{S^1} \mu(\mathrm{d}\theta)\, \frac{\sin(\eta-\theta)}{1+r^2-2r\cos(\eta-\theta)} \\ &= \lim_{\delta\downarrow 0} \frac{1}{2}\int_{|\eta-\theta|\geqslant\delta} \mu(\mathrm{d}\theta)\, \cot\left(\frac{\eta-\theta}{2}\right).\end{aligned} \tag{9}$$

The limits in (9) exist almost everywhere and for each $\epsilon > 0$ the set on which $|\mathscr{H}\mu|$ is larger than $1/\epsilon$ has Lebesgue measure less or equal to ϵ. We shall now express the current and asymptotic current in terms of G and thus in terms of the spectral properties of the trap, i.e., of the measure μ.

Lemma 3.1. With the notation of above

$$J(t) = 1 - \sum_{s=1}^{t-1} |K(s)|^2, \tag{10}$$

where the function K is determined by the relation

$$F(z) := \frac{G(z)}{1+G(z)} = \sum_{s=1}^{\infty} z^s K(s), \qquad |z| < 1. \tag{11}$$

The asymptotic current is given by

$$J_\infty = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_{S^1} \mathrm{d}\theta \, \frac{1 + 2\, \Re\mathrm{e}\, G(r e^{i\theta})}{|1 + G(r e^{i\theta})|^2}. \tag{12}$$

Proof. We introduce

$$K(t) := \langle (U^*)^t \varphi, (\mathbb{1} - P_{t-1}) \cdots (\mathbb{1} - P_1) \varphi \rangle$$

and denote by F the Z-transform of K

$$F(z) := \sum_{s=1}^{\infty} z^s K(s), \qquad |z| < 1. \tag{13}$$

The relation (10) follows from a straightforward computation

$$\begin{aligned} J(t) &= \|(\mathbb{1} - P_{t-1}) \cdots (\mathbb{1} - P_1) \varphi\|^2 \\ &= \|(\mathbb{1} - P_{t-2}) \cdots (\mathbb{1} - P_1) \varphi\|^2 \\ &\quad - |\langle (U^*)^{t-1} \varphi, (\mathbb{1} - P_{t-2}) \cdots (\mathbb{1} - P_1) \varphi \rangle|^2 \\ &= J(t-1) - |K(t-1)|^2. \end{aligned}$$

Next, we write

$$(\mathbb{1} - P_t) \cdots (\mathbb{1} - P_1) \varphi = (\mathbb{1} - P_{t-1}) \cdots (\mathbb{1} - P_1) \varphi - K(t)(U^*)^t \varphi.$$

Therefore

$$(\mathbb{1} - P_t) \cdots (\mathbb{1} - P_1) \varphi = \varphi - \sum_{s=1}^{t} K(s)(U^*)^s \varphi. \tag{14}$$

Taking the scalar product of $(U^*)^{t+1} \varphi$ with (14), we obtain

$$K(t+1) = \langle (U^*)^{t+1} \varphi, \varphi \rangle - \sum_{s=1}^{t} \langle (U^*)^{t-s+1} \varphi, \varphi \rangle \, K(s). \tag{15}$$

Equation (15) has the structure of a one-sided convolution equation. Using the Z-transform it becomes an algebraic equation. More precisely, multiplying (15) with z^{t+1} and summing from $t=0$ to ∞, we obtain

$$F(z) = G(z) - F(z)\, G(z).$$

As by (7) $1+G$ never vanishes inside the unit disk

$$\frac{G(z)}{1+G(z)} = F(z) = \sum_{s=1}^{\infty} z^s\, K(s),$$

proving (11).

The basic relation between the asymptotic current J_∞ and the measure μ is obtained by applying Parseval's formula to (13). Putting $z = r\mathrm{e}^{i\eta}$

$$\frac{1}{2\pi}\int_{S^1} \mathrm{d}\eta\, |F(r\mathrm{e}^{i\eta})|^2 = \sum_{s=1}^{\infty} r^{2s}\, |K(s)|^2. \tag{16}$$

We compute now the asymptotic current on the basis of (10)

$$\begin{aligned} J_\infty &= \lim_{t\to\infty} J(t) = 1 - \lim_{t\to\infty} \sum_{s=1}^{t-1} |K(s)|^2 \\ &= 1 - \lim_{r\uparrow 1} \sum_{s=1}^{\infty} r^{2s}\, |K(s)|^2 = 1 - \lim_{r\uparrow 1} \frac{1}{2\pi}\int_{S^1} \mathrm{d}\theta\, |F(r\mathrm{e}^{i\theta})|^2 \\ &= \lim_{r\uparrow 1} \frac{1}{2\pi}\int_{S^1} \mathrm{d}\theta\, \frac{1+2\,\Re\mathrm{e}\, G(r\mathrm{e}^{i\theta})}{|1+G(r\mathrm{e}^{i\theta})|^2}. \quad \blacksquare \end{aligned} \tag{17}$$

Our first result deals with the asymptotic current for trap states with absolutely continuous dynamical spectrum.

Theorem 3.1. Suppose that μ is absolutely continuous w.r.t. the Lebesgue measure, then $J_\infty > 0$.

Proof. Let μ be absolutely continuous with respect to the Lebesgue measure with density ρ. We have for $r < 1$

$$(1+2\,\Re\mathrm{e}\, G)(r\mathrm{e}^{i\eta}) = \frac{1}{2\pi}\int_{\mathcal{S}^1} \mathrm{d}\theta\, \rho(\theta+\eta)\, \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Because ρ is integrable and because of the properties of the Poisson kernel

$$\lim_{r\uparrow 1}\,(1+2\,\Re\mathrm{e}\,G)(r\mathrm{e}^{i\eta})=\rho(\eta)\quad\text{a.e.}$$

Also the imaginary part of

$$z\mapsto\frac{1}{2\pi}\int_{\mathscr{S}^1}\mathrm{d}\theta\;\rho(\theta)\,\frac{z}{\mathrm{e}^{i\theta}-z}$$

converges almost everywhere to the Hilbert transform $\mathscr{H}\rho$ of $\rho(\theta)\,\mathrm{d}\theta$:

$$\lim_{r\uparrow 1}\,\Im\mathrm{m}\,G(r\mathrm{e}^{i\eta})=(H\rho)(\eta):=\lim_{\delta\downarrow 0}\frac{1}{4\pi}\int_{|\theta|\geqslant\delta}\mathrm{d}\theta\;\rho(\theta+\eta)\,\cot\left(\frac{\theta}{2}\right).$$

In the expression for the asymptotic current

$$\begin{aligned}J_\infty&=\lim_{r\uparrow 1}\frac{1}{2\pi}\int_{\mathscr{S}^1}\mathrm{d}\theta\;(1-|F(r\mathrm{e}^{i\theta})|^2)\\&=\lim_{r\uparrow 1}\frac{1}{2\pi}\int_{\mathscr{S}^1}\mathrm{d}\theta\;\frac{1+2\,\Re\mathrm{e}\,G(r\mathrm{e}^{i\theta})}{1+2\,\Re\mathrm{e}\,G(r\mathrm{e}^{i\theta})+(\Re\mathrm{e}\,G)^2\,(r\mathrm{e}^{i\theta})+(\Im\mathrm{m}\,G)^2\,(r\mathrm{e}^{i\theta})}\end{aligned}$$

the integrand is bounded by 1 and tends almost everywhere to

$$\frac{4\rho(\theta)}{(1+\rho(\theta))^2+4(\mathscr{H}\rho)^2\,(\theta)}$$

as r grows to 1. We can therefore apply the dominated convergence theorem to obtain

$$J_\infty=\frac{1}{2\pi}\int_{\mathscr{S}^1}\mathrm{d}\theta\;\frac{4\rho(\theta)}{(1+\rho(\theta))^2+4(\mathscr{H}\rho)^2\,(\theta)}>0.\quad\blacksquare$$

In (17), we have replaced a limit $t\to\infty$ by a limit $r\uparrow 1$. In fact more information can be gained in doing so. Indeed, in the case $J_\infty=0$, the behaviour of $J(t)$ for large t is that of $1-\sum_{s=1}^{t-1}|K(s)|^2$. The function

$$\tilde{J}(r):=1-\sum_{s=1}^{\infty}r^{2s}\,|K(s)|^2,\qquad 0\leqslant r<1\tag{18}$$

is expressed in terms of the discrete Laplace transform of $s \mapsto |K(s)|^2$ and Tauberian theorems relate the large time behaviour of $t \mapsto J(t) = 1 - \sum_{s=1}^{t-1} |K(s)|^2$ with that of $\tilde{J}(r)$ when $r \uparrow 1$. We shall first use this idea to show that J_∞ vanishes when μ is singular. Next, we shall in a few examples consider convergence exponents.

Theorem 3.2. If μ is singular w.r.t. the Lebesgue measure, then $J_\infty = 0$.

Proof. Let μ be singular and hence concentrated on a measurable set with zero Lebesgue measure. As the Lebesgue measure of this set equals the infimum of the Lebesgue measures of open subsets containing the set, we can given any positive ϵ find an open subset A of $\mathscr{S}^1$ such that the Lebesgue measure of A is not larger than ϵ and $\mu(A) = 1$. The set A is a countable union of disjoint open intervals $]\alpha_j, \beta_j[$. We dress each of the $]\alpha_j, \beta_j[$ with open strips of width δ_j which shall be determined later on. By choosing the δ_j sufficiently small we may still ensure that the Lebesgue measure of $A^+ := \bigcup_j \,]\alpha_j - \delta_j, \beta_j + \delta_j[$ is small.

We shall also need

$$\begin{aligned} \frac{1}{2\pi} \int_{|\theta| \geqslant \delta} \mathrm{d}\theta \, \frac{1-r^2}{1+r^2-2r\cos\theta} &\leqslant 1 \qquad \text{for} \quad \delta \leqslant 1-r \\ &\leqslant \frac{1-r}{\delta} \qquad \text{for} \quad \delta \geqslant 1-r. \end{aligned} \tag{19}$$

The estimate for the case $\delta > 1-r$ is obtained as follows:

$$\begin{aligned} \frac{1}{2\pi} \int_{|\theta| \geqslant \delta} \mathrm{d}\theta \, \frac{1-r^2}{1+r^2-2r\cos\theta} &= \frac{1}{\pi} \int_\delta^\pi \mathrm{d}\theta \, \frac{1-r^2}{1+r^2-2r\cos\theta} \\ &\leqslant \frac{2(1-r)}{\pi} \int_\delta^\pi \mathrm{d}\theta \, \frac{1}{1+r^2-2r\cos\theta} \\ &= \frac{2(1-r)}{\pi} \int_\delta^\pi \mathrm{d}\theta \, \frac{1}{(1-r)^2+2r(1-\cos\theta)} \\ &\leqslant \frac{(1-r)}{\pi r} \int_\delta^\pi \mathrm{d}\theta \, \frac{1}{1-\cos\theta} \\ &\leqslant \frac{1-r}{\delta} \qquad \text{for } \delta \text{ sufficiently small.} \end{aligned}$$

We now estimate the current

$$
\begin{aligned}
\tilde{J}(r) &= \frac{1}{2\pi}\int_{\mathscr{S}^1} \mathrm{d}\eta (1-|F(re^{i\eta})|^2) \\
&= \frac{1}{2\pi}\int_{A^+} \mathrm{d}\eta (1-|F(re^{i\eta})|^2) + \frac{1}{2\pi}\int_{\mathscr{S}^1\setminus A^+} \mathrm{d}\eta (1-|F(re^{i\eta})|^2) \\
&\leqslant \epsilon + 2\sum_j \delta_j + \frac{1}{2\pi}\int_{\mathscr{S}^1\setminus A^+} \mathrm{d}\eta (1-|F(re^{i\eta})|^2) \\
&= \epsilon + 2\sum_j \delta_j + \frac{1}{2\pi}\int_{\mathscr{S}^1\setminus A^+} \mathrm{d}\eta \, \frac{1+2\,\Re\mathrm{e}\, G}{1+2\,\Re\mathrm{e}\, G + (\Re\mathrm{e}\, G)^2 + (\Im\mathrm{m}\, G)^2}(re^{i\eta}) \\
&\leqslant \epsilon + 2\sum_j \delta_j + \frac{1}{2\pi}\int_{\mathscr{S}^1\setminus A^+} \mathrm{d}\eta \, \frac{1+2\,\Re\mathrm{e}\, G}{1+2\,\Re\mathrm{e}\, G + (\Re\mathrm{e}\, G)^2}(re^{i\eta}) \\
&\leqslant \epsilon + 2\sum_j \delta_j + \frac{2}{\pi}\int_{\mathscr{S}^1\setminus A^+} \mathrm{d}\eta \int_{\mathscr{S}^1} \mu(\mathrm{d}\theta)\, \frac{1-r^2}{1+r^2-2r\cos(\eta-\theta)} \\
&= \epsilon + 2\sum_j \delta_j + \frac{2}{\pi}\int_{\mathscr{S}^1\setminus A^+} \mathrm{d}\eta \int_{\mathscr{S}^1} \mu(\mathrm{d}\theta)\, \delta_r(\eta-\theta) \\
&\leqslant \epsilon + 2\sum_j \delta_j + \frac{2}{\pi}\int_{A} \mu(\mathrm{d}\theta) \int_{\mathscr{S}^1\setminus A^+} \mathrm{d}\eta\, \delta_r(\eta-\theta).
\end{aligned} \tag{20}
$$

In the last but one inequality we have used that the integrand is bounded from above by $4(1+2\,\Re\mathrm{e}\, G)$. In the last inequality we use (19) to get

$$
\begin{aligned}
\tilde{J}(r) \leqslant \epsilon + 2\sum_j \delta_j &+ \frac{2}{\pi}\Bigg(\sum_{\substack{k \\ \delta_k < 1-r}} \mu(]\alpha_k, \beta_k[)\Bigg) \\
&+ \frac{2}{\pi}(1-r)\Bigg(\sum_{\substack{k \\ \delta_k \geqslant 1-r}} \frac{\mu(]\alpha_k, \beta_k[)}{\delta_k}\Bigg).
\end{aligned} \tag{21}
$$

For a given ϵ and a given large N, we may take $\delta_k = N\mu(]\alpha_k, \beta_k[)(1-r)$. When r is sufficiently close to 1, the upper bound for J becomes

$$
\begin{aligned}
\tilde{J}(r) \leqslant \epsilon + 2N(1-r) &+ \frac{2}{\pi}\Bigg(\sum_{\substack{k \\ N\mu(]\alpha_k, \beta_k[) < 1}} \mu(]\alpha_k, \beta_k[)\Bigg) \\
&+ \frac{2}{N\pi}\#(\{k \mid N\mu(]\alpha_k, \beta_k[) \geqslant 1\}).
\end{aligned} \tag{22}
$$

First we fix an arbitrary small ϵ and the corresponding sets $]\alpha_k, \beta_k[$. The second last term in (22) can be made small by choosing N sufficiently large. Also the last term becomes small because the nth term in a converging, monotonically decreasing, non-negative series is of order $o(1/n)$. Finally the second term in (22) becomes small when we let $r \uparrow 1$. ∎

We conclude this section with some examples, remarks and partial results about exponents. Let us assume that we are in the situation $J_\infty = 0$ and that we can assign an exponent to J, i.e.,

$$\gamma := \lim_{t \to \infty} -\frac{\log J(t)}{\log t}$$

exists. We shall, moreover, assume that infinitely many particles eventually are absorbed by the trap and actually strengthen this condition to $\gamma < 1$. We are interested in relating the behaviour of $t \mapsto J(t)$ as $t \to \infty$ with that of $r \mapsto \tilde{J}(r)$ as $r \uparrow 1$. We therefore introduce upper and lower exponents $\bar{\alpha}$ and $\underline{\alpha}$ for $\tilde{J}$

$$\bar{\alpha} := \limsup_{r \uparrow 1} \frac{\log \tilde{J}(r)}{\log(1-r)} \qquad \text{and} \qquad \underline{\alpha} := \liminf_{r \uparrow 1} \frac{\log \tilde{J}(r)}{\log(1-r)}.$$

Lemma 3.2. Using the notations and assumptions of above, $\bar{\alpha} = \underline{\alpha} = \gamma$.

Proof. We have to consider

$$J(t) = 1 - \sum_{s=1}^{t-1} |K(s)|^2 = \sum_{s=t}^{\infty} |K(s)|^2 \qquad \text{and}$$

$$\tilde{J}(r) = 1 - \sum_{s=1}^{\infty} r^{2s} |K(s)|^2 = \sum_{s=1}^{\infty} (1 - r^{2s})\, |K(s)|^2.$$

For notational convenience, we put $c_t := |K(t)|^2$ and $\lambda := -2 \log r$.

Let $\alpha > \gamma$, then

$$\begin{aligned}\lim_{\lambda \downarrow 0} \lambda^{-\alpha} \sum_{s=1}^{\infty} (1 - \mathrm{e}^{-\lambda s})\, c_s &\geqslant \lim_{\lambda \downarrow 0} \left(1 - \frac{1}{\mathrm{e}}\right) \lambda^{-\alpha} \sum_{s=[\lambda^{-1}]}^{\infty} c_s \\ &= \lim_{N \to \infty} \left(1 - \frac{1}{\mathrm{e}}\right) N^{\alpha} \sum_{s=N}^{\infty} c_s = \infty.\end{aligned}$$

Hence, $\underline{\alpha} \geqslant \gamma$.

Conversely, let $\alpha < \gamma$ and introduce the short notation $f(N) := \sum_{s=N}^{\infty} c_s$. Fix an arbitrary $\epsilon > 0$ and a γ_0 such that $\alpha < \gamma_0 < \gamma$. For $\lambda > 0$ and $\kappa = 2, 3, \ldots$, let $\tilde{N}_n$ be determined by

$$\exp(-\lambda \tilde{N}_n) = \frac{\kappa - n}{\kappa}, \qquad n = 1, 2, \ldots, \kappa - 1$$

and put $N_n := [\tilde{N}_n]^+$ where $[x]^+$ is the smallest integer larger or equal to x. We shall pick κ later on in such a way that $\kappa \ll \lambda^{-1}$. This implies that the N_j are far apart, in particular that $N_1 \gg 1$. We now have obtained a partition

$$1 =: N_0 \ll N_1 \ll \cdots \ll N_{\kappa-1} \ll N_\kappa := +\infty$$

of $\mathbb{N}_0$ such that

$$1 - \mathrm{e}^{-\lambda t} \leqslant \frac{n+1}{\kappa} \qquad \text{for} \quad N_n \leqslant t < N_{n+1}.$$

$$\begin{aligned}
\sum_{s=1}^{\infty} (1 - \mathrm{e}^{-\lambda s})\, c_s &= \sum_{n=0}^{\kappa-1} \sum_{s=N_n}^{N_{n+1}-1} (1 - \mathrm{e}^{-\lambda s})\, c_s \\
&\leqslant \sum_{n=0}^{\kappa-1} \frac{n+1}{\kappa} \sum_{s=N_n}^{N_{n+1}-1} c_s \\
&= \sum_{n=0}^{\kappa-1} \frac{n+1}{\kappa} (f(N_n) - f(N_{n+1})) \\
&\leqslant \frac{1}{\kappa} (f(N_0) + f(N_1) + \cdots + f(N_{\kappa-1})) \\
&\leqslant \frac{1}{\kappa} (1 + \epsilon N_1^{-\gamma_0} + \cdots + \epsilon N_{\kappa-1}^{-\gamma_0}).
\end{aligned} \tag{23}$$

The inequality in (23) holds for N_1 large enough, i.e., for κ sufficiently large. Because $\tilde{N}_n = [\frac{1}{\lambda} \log(\frac{\kappa}{\kappa - n})]^+$ we have

$$N_n^{-\gamma_0} < \left(\frac{1}{\lambda} \log \left(\frac{1}{1 - \frac{n}{\kappa}} \right) \right)^{-\gamma_0}$$

and thus

$$\begin{aligned}
\sum_{s=1}^{\infty}(1-\mathrm{e}^{-\lambda s})\,c_s &\leqslant \frac{1}{\kappa}+\frac{\epsilon}{\kappa}\sum_{n=1}^{\kappa-1}\left(\frac{1}{\lambda}\log\left(\frac{1}{1-\frac{n}{\kappa}}\right)\right)^{-\gamma_0}\\
&\leqslant \frac{1}{\kappa}+\lambda^{\gamma_0}\frac{\epsilon}{\kappa}\sum_{n=1}^{\kappa-1}\left(\log\left(\frac{1}{1-\frac{n}{\kappa}}\right)\right)^{-\gamma_0}\\
&\leqslant \frac{1}{\kappa}+\lambda^{\gamma_0}\epsilon\int_0^1 \mathrm{d}y\left(\log\left(\frac{1}{1-y}\right)\right)^{-\gamma_0}\\
&= \frac{1}{\kappa}+\lambda^{\gamma_0}\epsilon\int_0^1 \mathrm{d}t\left(\log\frac{1}{t}\right)^{-\gamma_0}\\
&= \frac{1}{\kappa}+\lambda^{\gamma_0}\epsilon\,\Gamma(1-\gamma_0)\\
&\leqslant \delta\lambda^{\gamma_0}
\end{aligned}$$

with δ arbitrarily small. This last inequality is obtained by choosing $\lambda^{-\gamma_0} \ll \kappa \ll \lambda^{-1}$ which is possible because $0 \leqslant \gamma_0 < 1$. Hence $\bar{\alpha} \leqslant \gamma$ and the lemma is proven. ▮

When estimating the current in the proof of Theorem 3.2 we dropped the contribution of $\mathfrak{Im}\, G$. Generally, this may lead to underestimate the exponent. A simple example is provided by a measure μ that is concentrated on a finite set such as $\mu(\mathrm{d}\theta)=\delta(\theta)\,\mathrm{d}\theta$ (the general case being quite similar). A simple calculation shows that $F(z)=z$. Therefore

$$\frac{1}{2\pi}\int_{\mathscr{S}^1}\mathrm{d}\eta(1-|F(r\mathrm{e}^{i\eta})|^2)=1-r^2\sim 2(1-r)$$

and the true exponent is 1. Dropping the imaginary part of G in the integral, we get an exponent $1/2$:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathscr{S}^1}\mathrm{d}\eta\,\frac{1+2\,\mathfrak{Re}\,G}{1+2\,\mathfrak{Re}\,G+(\mathfrak{Re}\,G)^2}(r\mathrm{e}^{i\eta}) &= \frac{1}{2\pi}\int_{\mathscr{S}^1}\mathrm{d}\eta\,\frac{(1-r^2)(1+r^2-2r\cos\eta)}{(1-r\cos\eta)^2}\\
&\sim \int \mathrm{d}x\,\frac{(1-r)\,x^2}{(2(1-r)+x^2)^2}\\
&\sim \sqrt{1-r}.
\end{aligned}$$

Let μ be a discrete measure, possibly concentrated on a dense subset of $\mathscr{S}^1$,

$$\mu(\mathrm{d}\theta) = \sum_j \rho_j \delta(\theta - \theta_j)\, \mathrm{d}\theta$$

with

$$\rho_j \geqslant 0, \quad \sum_j \rho_j = 1 \quad \text{and} \quad i \neq j \Rightarrow \theta_i \neq \theta_j.$$

Applying the estimates in the proof of Theorem 3.2 we obtain

$$\tilde{J}(r) \leqslant 2 \sum_j \delta_j + \frac{2}{\pi} \sum_{\substack{j \\ \delta_j \geqslant 1-r}} \frac{\rho_j(1-r)}{\delta_j} + \frac{2}{\pi} \sum_{\substack{j \\ \delta_j \leqslant 1-r}} \rho_j. \tag{24}$$

Suppose that the ρ_j tend sufficiently rapidly to zero in order that also $\sum_j \sqrt{\rho_j} < \infty$. Choosing $\delta_j = \sqrt{(1-r)\, \rho_j}$, we obtain

$$\tilde{J}(r) \leqslant 2 \left(1 + \frac{1}{\pi}\right) \left(\sum_j \sqrt{\rho_j}\right) \sqrt{1-r}$$

and therefore an exponent $1/2$. In view of the first example, this is the best exponent we can hope to obtain neglecting the contribution of the imaginary part of G.

Suppose that $\rho_j \sim j^{-\alpha}$ with $\alpha > 1$. Choosing in (24) $\delta_j = N\rho_j(1-r)$ with N large, we obtain

$$\tilde{J}(r) \leqslant 2N(1-r) + \frac{2}{N\pi} \# \left(\left\{ j \,\middle|\, \rho_j \geqslant \frac{1}{N} \right\} \right) + \frac{2}{\pi} \sum_{\substack{j \\ \rho_j \leqslant \frac{1}{N}}} \rho_j.$$

The optimal choice for N is $N \sim (1-r)^{-\alpha/(2\alpha - 1)}$ and this yields an exponent $(\alpha - 1)/(2\alpha - 1) < 1/2$. The exponent $1/2$ is reached for all ρ_j that tend faster to zero that any inverse power of j.

In our last example the measure μ will be singular continuous.[12] Given a number $x \in [0, 1]$, we write its binary expansion as

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{2^m},$$

where $a_m(x) \in \{0, 1\}$ for $m \geqslant 1$. This defines a map $F: \{0, 1\}^{\mathbb{N}} \to [0, 1]$ which can be used to transport a measure λ on $\{0, 1\}^{\mathbb{N}}$ to a measure μ on

$[0, 1]$ by the relation $\mu(A) = \lambda(F^{-1}[A])$. Take for λ the infinite product of the measure $(1-p, p)$, $p \in [0, 1]$. The corresponding measure μ on $[0, 1]$ is called the Bernoulli measure μ_p. Except for $p=0$, $p=1$ (Dirac measures) and $p=1/2$ (Lebesgue measure), μ_p is singular continuous.

We want to estimate the exponent α of the current $\tilde{J}(r) \sim (1-r)^{\alpha}$. Assume $0 < p < 1/2$ and let q be a number between p and $1/2$. Define the sets $A(n)$ for $n \geqslant 1$ as

$$A(n) = \{x \in [0, 1] \mid \#\{m \leqslant n \mid a_m(x) = 1\} < qn\}.$$

The asymptotic behaviour of the Bernoulli measure is

$$1 - \mu_p(A(n)) \sim \exp(-nS(q \mid p)),$$

whereas for the Lebesgue measure

$$|A(n)| \sim \exp(-nS(q \mid 1/2)),$$

with $S(p_1 \mid p_2)$ the relative entropy of the probability measures $(p_1, 1-p_1)$ with respect to $(p_2, 1-p_2)$, i.e.,

$$\begin{aligned} S(p_1 \mid p_2) := {} & p_1 \log(p_1) + (1-p_1) \log(1-p_1) \\ & - p_1 \log(p_2) - (1-p_1) \log(1-p_2). \end{aligned}$$

The sets $A(n)$ are finite unions of $K(n)$ intervals $A_k(n)$ for which the first n digits of the binary expansion are given. We have the asymptotics

$$K(n) \sim \exp(nS(q)),$$

where $S(q)$ is the Shannon entropy of the probability measure $(q, 1-q)$, i.e.,

$$S(q) := -q \log(q) - (1-q) \log(1-q).$$

As in the proof of Theorem 3.2 we dress the intervals $A_k(n)$ by small strips of length $\delta_k(n)$. This results in the set $A^+(n)$. We can then use the estimate (20) for the current $\tilde{J}(r)$

$$\begin{aligned} & \int_{S \setminus A^+(n)} \mathrm{d}\eta \left\{ \int_{A(n)} \mu_p(\mathrm{d}\theta)\, \delta_r(\eta - \theta) + \int_{S \setminus A(n)} \mu_p(\mathrm{d}\theta)\, \delta_r(\eta - \theta) \right\} \\ & \leqslant \frac{2}{\pi} \sum_{\substack{k \\ \delta_k(n) < 1-r}} \mu_p(A_k(n)) + (1-r) \sum_{\substack{k \\ \delta_k(n) > 1-r}} \frac{\mu_p(A_k(n))}{\delta_k} + (1 - \mu_p(A(n))). \end{aligned}$$

Again taking $\delta_k(n) = N\mu_p(A_k(n))(1-r)$, we obtain

$$\tilde{J}(r) \leqslant |A(n)| + 2N(1-r) + \sum_{\substack{k \\ N\mu_p(A_k(n))<1}} \mu_p(A_k(n))$$
$$+\frac{1}{N}\#\{k \mid N\mu_p(A_k(n)) > 1\} + (1-\mu_p(A(n))).$$

Imposing now the scaling behaviour $N \sim \exp(n\nu)$ and $1-r \sim \exp(-n\beta)$, we find

$$\alpha \geqslant \left\{\frac{\min\{S(q \mid 1/2),\, \beta-\nu,\, \nu-S(q),\, S(q \mid p)\}}{\beta}\right\}$$

for any q, ν, and β. The optimal values are

$$q = \frac{\log 2(1-p)}{\log \frac{p}{1-p}}$$

(for which $S(q \mid 1/2) = S(q \mid p)$), $\nu = \log 2$ and $\beta = 2\log 2 - S(q)$. Finally, the lower bound for the exponent α is

$$\alpha \geqslant \frac{\log 2 - S(q)}{2\log 2 - S(q)}. \tag{25}$$

We compare this bound with some numerical computations. For a few pairs (p, r) the integrals for $G(z)$ and $\tilde{J}(r)$ were evaluated on an equidistant mesh of 2^{13} points. To illustrate the approximation made in (20) by dropping $\mathfrak{Im}\, G$, we performed the computation both with and without this imaginary part. The relative precision for the current $\tilde{J}(r)$ was checked to be better than 0.01, while for the exponent α it is 0.1. Figure 1 shows the

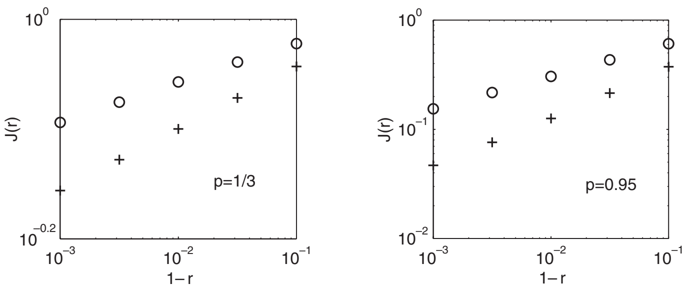

Fig. 1. The current $\tilde{J}(r)$, with (crosses) and without (circles) the contribution $\mathfrak{Im}\, G$. Left, $p = 1/3$ and right, $p = 0.95$.

Table I. Estimates for the Exponent α

	analytical	numerical without $\mathfrak{Im}\, G$	numerical with $\mathfrak{Im}\, G$
$p=1/3$	2.05×10^{-2}	3.7×10^{-2}	5.6×10^{-2}
$p=0.95$	1.96×10^{-1}	3.2×10^{-1}	4.2×10^{-1}

existence of the exponents and the error introduced by neglecting the imaginary part of $G(z)$. The latter is quantified in Table I. For this example the lower bound (25) for α is rather sharp.

4. DYNAMICAL ENTROPY OF A FERMION DYNAMICS

We shall in this section apply our results in the setting of a free Fermionic gas. As this is a system of non-interacting particles, it is completely described in terms of single-particle quantities. Second quantisation allows to lift one-particle objects to the many particles, taking into account the Fermi statistics. We remind here briefly the mathematical setup.

We shall denote the single-particle Hilbert space by $\mathfrak{H}$. The observables of the Fermion algebra, also called CAR for canonical anticommutation relations, is the C^*-algebra $\mathscr{A}(\mathfrak{H})$ determined through the relations

$$a(f+\alpha g)=a(f)+\bar{\alpha}a(g),\ \ \{a(f), a(g)\}=0,\ \text{ and }\ \{a(f), a^*(g)\}=\langle f, g\rangle.$$

Sometimes we shall deal with a one-particle space $\mathfrak{H}$ of finite dimension d. In this case $\mathscr{A}(\mathfrak{H})$ is easily seen to be isomorphic to the algebra of matrices of dimension 2^d. An explicit construction is given in terms of linear transformations of the antisymmetric Fock space $\Gamma(\mathfrak{H})$ which is spanned by the n-particle vectors

$$a^*(f_1)\, a^*(f_2)\cdots a^*(f_n)\,\Omega$$

for $0\leqslant n\leqslant d$. The normalised vector Ω is called the vacuum and it is annihilated by any operator $a(f)$.

The construction of dynamical entropy of a conservative evolution Θ presented in refs. 13 and 14 is based on the following idea. Given a unital C^*-algebra $\mathfrak{A}$ and a reference state ω, one considers an operational partition, i.e., a finite collection $\mathfrak{X}=\{x_1, x_2,\ldots, x_n\}$ of elements of $\mathfrak{A}$ satisfying $\sum_j x_j^* x_j=\mathbb{1}$. This yields a correlation matrix

$$\rho[\mathfrak{X}]:=[\omega(x_j^* x_i)]$$

with corresponding von Neumann entropy

$$\mathsf{H}[\omega, \mathfrak{X}] := \operatorname{Tr} \eta(\rho[\mathfrak{X}]),$$

where η is the usual entropy function

$$\eta(0) := 0 \quad \text{and} \quad \eta(x) := -x \log x, \qquad 0 < x \leqslant 1.$$

The average entropy of an operational partition $\mathfrak{X} = (x_1, x_2, \ldots, x_n)$ arises by computing the average entropy of the correlation matrix $\mathfrak{X}_t$ corresponding to the refinement of the partition $\mathfrak{X}$ at discrete times up to $t-1$. More precisely

$$\mathfrak{X}_t := \Theta^{t-1}(\mathfrak{X}) \circ \cdots \circ \Theta(\mathfrak{X}) \circ \mathfrak{X}.$$

In this expression,

$$\Theta^s(\mathfrak{X}) := (\Theta^s(x_1), \Theta^s(x_2), \ldots, \Theta^s(x_n)) \qquad \text{and}$$
$$\mathfrak{X} \circ \mathfrak{Y} := (x_1 y_1, x_2 y_1, \ldots, x_n y_m) \qquad \text{for} \quad \mathfrak{Y} = (y_1, y_2, \ldots, y_m).$$

It can happen that the growth of $t \mapsto \mathsf{H}[\omega, \mathfrak{X}_t]$ is sublinear if the dynamics is not sufficiently randomising. In such a case one can look for a growth exponent.

We shall in the following pages obtain a lower bound for $\mathsf{H}[\omega, \mathfrak{X}_t]$ in terms of particle numbers absorbed by a trap for the case of a weakly interacting Fermion system, meaning that we may use an effective one-particle dynamics $\Theta(a(f)) := a(Uf)$ for the evolution. U is a unitary operator on the one-particle space $\mathfrak{H}$. The trap will in fact define a particular partition of unity and the evolution of this partition will be linked to the evolution of the trap states which determines in turn the number of particles absorbed by the trap.

The reference state will be chosen accordingly as a gauge-invariant quasi-free state. Such a state ω_Q is uniquely determined by its symbol Q which is a linear operator on $\mathfrak{H}$ satisfying $0 \leqslant Q \leqslant \mathbb{1}$. The only monomials in the creation and annihilation operators a^* and a which have non-zero expectations contain a same number of each and

$$\omega_Q(a^*(f_1) \cdots a^*(f_n)\, a(g_n) \cdots a(g_1)) = \det(\langle g_i, Q f_j \rangle).$$

In particular, we may choose $Q = \kappa \mathbb{1}$ for $0 \leqslant \kappa \leqslant 1$. Such states are homogeneous and in, e.g., the case of Fermions on a lattice they describe independent Fermions occupying each site of the lattice with probability κ. The

Fock vacuum, i.e., the vector state determined by Ω in the Fock representation of above, corresponds to the choice $\kappa = 0$. The choice $\kappa = 1/2$ corresponds to the unique tracial state on $\mathscr{A}(\mathfrak{H})$. A quasi-free state ω_Q is known to be pure if and only if Q is an orthogonal projector. Moreover, any ω_Q can be obtained as the restriction of a pure quasi-free state on a larger CAR algebra by using the purification construction. One introduces the auxiliary space $\mathfrak{K} := \overline{Q(\mathbb{1} - Q)\,\mathfrak{H}}$ and the projection operator

$$\begin{pmatrix} Q & \sqrt{Q(\mathbb{1}-Q)}|_{\mathfrak{K}} \\ \sqrt{Q(\mathbb{1}-Q)} & (\mathbb{1}-Q)|_{\mathfrak{K}} \end{pmatrix} \tag{26}$$

on $\mathfrak{H} \oplus \mathfrak{K}$. For the homogeneous states $Q = \kappa\mathbb{1}$ with $0 < \kappa < 1$, $\mathfrak{K} = \mathfrak{H}$ and the projector becomes

$$\begin{pmatrix} \kappa\mathbb{1} & \sqrt{\kappa(1-\kappa)}\,\mathbb{1} \\ \sqrt{\kappa(1-\kappa)}\,\mathbb{1} & (1-\kappa)\,\mathbb{1} \end{pmatrix}.$$

For a symbol Q of finite rank, the entropy of ω_Q is given by

$$S(\omega_Q) = \mathrm{Tr}(\eta(Q) + \eta(\mathbb{1} - Q)). \tag{27}$$

The formula can obviously be extended to compact Q with eigenvalues converging sufficiently fast to 0.

In order to compute the dynamical entropy for quasi-free evolutions with a quasi-free reference state, it suffices to consider a restricted class of partitions $\mathscr{X}$ characterised by the property that

$$y \mapsto \Lambda(y) := \sum_j x_j^* \, y \, x_j$$

transforms the gauge-invariant quasi-free states into themselves. For an outline of the argument we refer to ref. 14. Such maps Λ are called gauge-invariant quasi-free completely positive maps and are determined by two linear operators V and W on $\mathfrak{H}$ obeying the restrictions

$$0 \leqslant W \leqslant \mathbb{1} - V^* V.$$

On a monomial Λ acts as

$$\Lambda(a^\#(f_1) \cdots a^\#(f_n)) = \sum_{S \subset \{1,\dots,n\}} \epsilon(S) \left(\prod_{j \in S} a^\#(V f_j) \right) \omega_W \left(\prod_{k \notin S} a^\#(f_k) \right).$$

In this formula, $a^\#$ denotes either a or a^* and $\epsilon(S)$ equals ± 1 according to the parity of the permutation defined by S. The quasi-free state ω_Q transforms under Λ into the quasi-free state with symbol

$$V^* Q V + W. \tag{28}$$

Even if different partitions may yield the same map Λ, $\mathsf{H}[\omega_Q, \mathfrak{X}]$ will only depend on Q and Λ and we shall derive in the next proposition its expression directly in terms of Q and Λ.

Proposition 4.1. Let $\dim(\mathfrak{H}) < \infty$ and let $\mathfrak{X} = (x_1, x_2, \dots, x_n)$ be an operational partition in $\mathscr{A}(\mathfrak{H})$ such that $y \mapsto \sum_j x_j^* y x_j$ is gauge-invariant quasi-free determined by (V, W). Let the symbol Q determine the gauge-invariant quasi-free state ω_Q, then

$$\mathsf{H}[\omega_Q, \mathfrak{X}] = S(\omega_R) \tag{29}$$

where R is the symbol on $\mathfrak{H} \oplus (Q(\mathbb{1} - Q)\, \mathfrak{H}) =: \mathfrak{H} \oplus \mathfrak{K}$ given by

$$R = \begin{pmatrix} V^* Q V + W & V^* \sqrt{Q(\mathbb{1} - Q)}|_{\mathfrak{K}} \\ \sqrt{Q(\mathbb{1} - Q)}\, V & (\mathbb{1} - Q)|_{\mathfrak{K}} \end{pmatrix}. \tag{30}$$

Proof. Let us denote by $(\mathfrak{H}_Q, \pi_Q, \Omega_Q)$ the GNS triple of ω_Q and by $(e_1, e_2, \dots, e_n)$ the canonical orthonormal basis of $\mathbb{C}^n$. The pure state on $\mathfrak{B}(\mathfrak{H}_Q) \otimes \mathscr{M}_n$ induced by the vector $\sum_j \pi_Q(x_j)\, \Omega_Q \otimes e_j \in \mathfrak{H}_Q \otimes \mathbb{C}^n$ restricts to generally mixed states on $\mathscr{M}_n$ and $\mathfrak{B}(\mathfrak{H}_Q)$ that have, up to multiplicities of 0, the same spectrum. A straightforward computation shows that this restriction to $\mathscr{M}_n$ is the correlation matrix $\rho[\mathfrak{X}]$ and that to $\mathfrak{B}(\mathfrak{H}_Q)$ the density matrix

$$\sum_j |\pi_Q(x_j)\, \Omega_Q\rangle\langle \pi_Q(x_j)\, \Omega_Q|.$$

As $\mathscr{A}(\mathfrak{H})$ is isomorphic to the algebra $\mathscr{M}_{2^d}$ where d is the dimension of the one-particle space $\mathfrak{H}$, we can write that $\mathfrak{H}_Q = \mathbb{C}^{2^d} \otimes \mathfrak{L}$ with $\pi_Q(x) = x \otimes \mathbb{1}_{\mathfrak{L}}$. Therefore, there exists a unique unity preserving completely positive map Γ on $\mathfrak{B}(\mathfrak{H}_Q)$ determined by the requirement

$$\Gamma(y \otimes z) := \Bigl(\sum_j \pi_Q(x_j^*)\, y \pi_Q(x_j)\Bigr) \otimes z, \qquad y \in \pi_Q(\mathscr{A}(\mathfrak{H})), \quad z \in \mathfrak{B}(\mathfrak{L}).$$

Obviously $\mathsf{H}[\omega_Q, \mathfrak{X}] = S(|\Omega_Q\rangle\langle\Omega_Q| \circ \Gamma)$. It now remains to compute this quantity.

Using the purification (26) we see that the GNS representation space of ω_Q is the Fock space built on $\mathfrak{H} \oplus (Q(\mathbb{1}-Q)\,\mathfrak{H})$ and that Γ is determined by the operators $(V \oplus \mathbb{1}, W \oplus 0)$. It suffices now to use formulas (28) and (27) to finish the proof. ∎

We shall now obtain a lower bound on the dynamical entropy in terms of currents of particles falling into a trap. This will generally not provide the optimal lower bound but we expect it to provide the correct growth exponent, which it certainly does in the case of linear growth. The second quantised version of a localised trap is provided by a quasi-free completely positive map Λ with operators $(V, 0)$. In order to compute the number Δ of particles that disappear from a homogeneous state ω_κ in the trap, we consider the particle number operator N in $\mathscr{A}(\mathfrak{H})$. $N := \sum_j a^*(e_j)\, a(e_j)$ where $\{e_1, e_2, \ldots\}$ is an orthonormal basis of $\mathfrak{H}$. We assume for the moment that $\mathfrak{H}$ is finite dimensional but the general case can be obtained by a suitable limiting procedure. Then

$$\Delta = \omega_\kappa(N - \Lambda(N)) = \kappa\, \mathrm{Tr}(\mathbb{1} - V^*V).$$

The locality of the trap is expressed by the condition

$$\mathrm{Rank}(\mathbb{1} - V^*V) < \infty, \qquad V^*V \leqslant \mathbb{1}.$$

In our case, all refined and evolved partitions remain quasi-free with strictly local action. An explicit computation shows that $\mathfrak{X}_t$ is determined by $(V_t, 0)$ with

$$V_t = [VU]^t\, U^{-t}.$$

Using the explicit expression of the entropy of correlation matrix (29) in terms of its symbol (30), we have

$$\mathsf{H}[\omega_\kappa, \mathfrak{X}_t] = \mathrm{Tr}(\eta(R_t) + \eta(\mathbb{1} - R_t)) \tag{31}$$

with

$$R_t = \begin{pmatrix} \kappa V_t^* V_t & \sqrt{\kappa(1-\kappa)}\, V_t^* \\ \sqrt{\kappa(1-\kappa)}\, V_t & 1-\kappa \end{pmatrix}. \tag{32}$$

In order to avoid a trivial situation we assume that $0<\kappa<1$, in which case the trace in (31) is taken over a space of dimension twice the rank of $\mathbb{1}-V_t^*V_t$. We write

$$\begin{pmatrix} \kappa V_t^* V_t & \sqrt{\kappa(1-\kappa)}\, V_t^* \\ \sqrt{\kappa(1-\kappa)}\, V_t & 1-\kappa \end{pmatrix} = \begin{pmatrix} \sqrt{\kappa}\, V_t^* & 0 \\ \sqrt{1-\kappa} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\kappa}\, V_t & \sqrt{1-\kappa} \\ 0 & 0 \end{pmatrix} =: A^* A.$$

But then

$$A\, A^* = \begin{pmatrix} 1-\kappa+\kappa V_t\, V_t^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Using that, up to multiplicities of 0, A^*A and $A\,A^*$ have the same spectrum and that $\eta(0)=\eta(1)=0$, the entropy becomes

$$\mathsf{H}[\omega_\kappa, \mathfrak{X}_t] = \mathrm{Tr}[\eta(1-\kappa(1-V_t^*V_t)) + \eta(\kappa(1-V_t^*V_t))].$$

Finally, as η is concave we obtain the lower bound

Proposition 4.2.

$$\mathsf{H}[\omega_\kappa, \mathfrak{X}_t] \geqslant \{\eta(\kappa)+\eta(1-\kappa)\}\ \mathrm{Tr}(\mathbb{1}-V_t^*V_t).$$

The usual computation of dynamical entropy involves two more steps. First the computation of the asymptotic rate of entropy production $\mathsf{h}[\omega, \mathfrak{X}]$ which consists in taking the limit for $t\to\infty$ of $\mathsf{H}[\omega, \mathfrak{X}_t]/t$ and next taking the supremum over a suitable class of operational partitions. Proposition 4.2 is more general in the sense that it provides a lower bound for $\mathsf{H}[\omega, \mathfrak{X}_t]$ even when this quantity scales in a sublinear way in t. We can also use the lower bound of the proposition to show that the dynamical entropy is strictly positive whenever there is a non-zero asymptotic current for a trap belonging to the class of allowed partitions. We have seen that $\kappa\, \mathrm{Tr}(\mathbb{1}-V_t^*V_t)$ represents the total amount of particles that disappeared from the homogeneous state ω_κ into the trap up to time t. The corresponding current at time t is then

$$\kappa\, \mathrm{Tr}(V_t^*V_t - V_{t-1}^*V_{t-1}),$$

which is precisely the quantity considered in Section 3. In particular, the entropy grows linearly in time if the absolutely continuous spectral subspace of the single-step unitary U is non-trivial. But even if U has no absolutely continuous spectral component, an estimate of the growth exponent of the entropy may be obtained.

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